Stochastic Approach to Modeling Fatigue Crack Growth

B. F. Spencer Jr.* and J. Tang†
University of Notre Dame, Notre Dame, Indiana
and
M. E. Artley‡
U.S. Air Force Wright Aeronautical Laboratories, Wright-Patterson Air Force Base, Ohio

An extensive set of fatigue crack growth (FCG) data known as the Virkler data is employed to investigate a recently developed stochastic FCG model that is based on concepts from both fracture mechanics and random process theory. The model represents the crack state as a two-dimensional vectored Markov process and, as a result, has eliminated many objections to previous stochastic FCG models. Two FCG laws are considered for use in the stochastic analysis and a modified finite-integral method (MFI) that does not require differentiation of the FCG data is suggested for determining the parameters of the stochastic FCG model. Excellent comparison between theoretical results and experimental data is obtained for all cases considered.

Introduction

SLOW crack growth is a major factor that must be considered in the design of durable airframes. A survey of U.S. Air Force maintenance records revealed great consistency in the frequency with which structural deficiencies occurred over all weapon systems.[2] In a relative numerical ranking of incidents, cracking placed first over corrosion, maintenance/manufacturing damage, and other reported incidents. From this, one can conclude that fatigue crack growth (FCG) is a primary damage mechanism to be reckoned with when insuring the durability of metallic airframes.

Within the last 15 years, an initial flaw assumption for durability has been adopted as a fatigue design specification. The USAF Military Specification for Aircraft Structures, MIL-A-87221,[3] requires that adverse cracking should not occur within two lifetimes of the airframe for a specified use and environment. The following guidance has been given for performing the related durability analysis: "the airframe should be designed such that assumed initial cracks in typical quality structure would not reach sizes to preclude repair wihtin the two service lifetimes." The initial flaw size assumptions are given in the specifications for a deterministic analysis.

A shortcoming in the above-stated specifications is its deterministic viewpoint. Consider the Virkler[1] data depicted in Fig. 1. These data are from 68 identical 2.54-mm-thick center cracked specimens cut from the same plate of 2024-T3 aluminum alloy. Each test was performed under identical loading and the crack length was recorded from 9.00 mm to a final length of 49.80 mm. Tightly controlled laboratory conditions were present in this test, yet significant scatter is seen in FCG data. Thus, it becomes apparent that deterministic models are inadequate for predicting the safety of a fatigue-critical structure. A probabilistic viewpoint is allowed by the USAF Military Specification in a subtle way. Alternate flaw shapes and sizes can be used when based on an equivalent initial flaw size (EIFS) approach.[4] This approach must currently be pursued if a reliability analysis is to be performed under the USAF Military Specification. Guidelines for performing a probabilistic durability analysis involving the EIFS approach are presented in the Durability Handbook.[5]

However, in the design and life prediction of fatigue-critical structures, one must account for the variability in the crack growth rate as well as in the initial quality of the material via initial flaw size distibutions. Although many life prediction methods exist with which to perform crack growth analysis deterministically and that can readily incorporate the EIFS approach (see, for example, Refs. 5 and 6 and the references contained therein), only a few exist that consider the stochastic nature of crack growth.[7-12] In this paper, a general stochastic approach is examined for modeling FCG in metals. In addition to the model accounting for variations in the crack growth rate, assumptions on an initial flaw size distribution are readily incorporated into the analysis. An extensive set of crack growth data obtained by Virkler et al.[1] will be employed herein to investigate the quality of this model for prediction of fatigue life. Finally, a modified finite-integral (MFI) approach is introduced to estimate several of the parameters used in the stochastic model.

Background

A serious impediment in the consideration of slow crack growth is the lack of an applicable stochastic FCG model. The key to a successful analysis is to model the crack growth rate accurately in conjunction with an accurate description of the initial quality of the structure. This approach would provide highly accurate life predictions for fatigue-critical structures. Here we are concerned with the former of the two aspects of this problem—stochastic FCG modeling.

Deterministic models that have been suggested based upon the principles of fracture mechanics, but that do not account for this variability, are generally of the form[6]

$$\frac{\mathrm{d}a(t)}{\mathrm{d}t} = Q(\Delta K, K_{\max}, S, a, R) \tag{1}$$

where $a(t)$ is the deterministic crack size, t the time or number of cycles, Q a non-negative function, ΔK the stress intensity factor range, $K_{\max}$ the maximum stress intensity factor, S the stress amplitude, and R the stress ratio. For deterministic loadings, Eq. (1) can be randomized (see Refs. 8, 9, and 11–21) to account for the inherent variability in crack growth rate by employing a multiplicative random process and reads

Presented as Paper 88-2419 at the AIAA 26th Structures, Structural Dynamics and Materials Meeting, Williamsburg, VA, April 18–20, 1988; received April 7, 1988; revision received Dec. 2, 1988.

*Assistant Professor, Department of Civil Engineering. Member AIAA.

†Graduate Assistant and Doctoral Candidate, Department of Civil Engineering.

‡Aerospace Engineer. Flight Dynamics Laboratory.

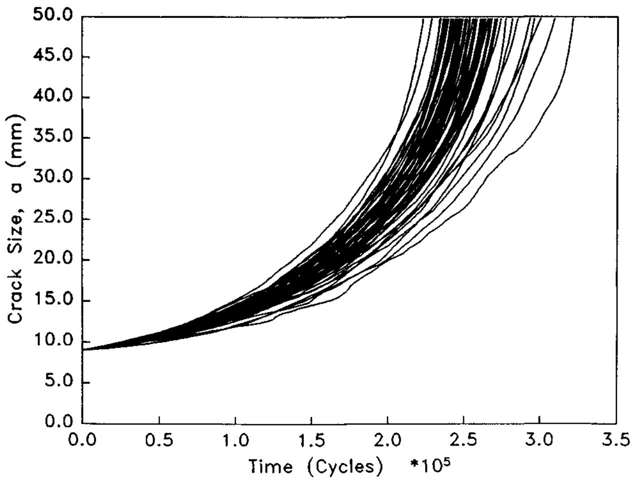

Fig. 1 Crack growth curves for 68 replicate tests observed by Virkler et al.[1]

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = Q(\Delta K, K_{\max}, S, A, R)X(t) \tag{2}$$

where $A(t)$ is the random process modeling the random crack size and $X(t)$ is required to be non-negative in order to eliminate the possibility of negative crack growth rates. In addition, upon examination of Fig. 1, we see that the sample a vs t paths cross, thus dictating that $X(t)$ also be a random process. While a number of researchers have studied Eq. (2), the only general analysis technique that has proved successful without resorting to approximation has been Monte Carlo simulation. Other approximate analytical solution techniques have had to resort to stochastic averaging, inconsistency in boundary conditions, or admission of negative crack growth rates in an effort to obtain solutions to this problem. The method advocated herein augments the FCG law in Eq. (2) with an auxiliary equation in order to overcome the objections raised concerning previous analyses. This approach allows for direct solutions of some of the previously unavailable statistics needed for the design of fatigue-critical structures such as those discussed above.

Two-State Markov Process

In order to model the history dependence of FCG, the random deviate $X(t)$ will be modeled by a general class of processes obtained through a transformation of an exponentially correlated Gaussian random process. Consider the Langevin equation, a first-order stochastic differential equation given by

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = -\xi Z + G(t) \tag{3}$$

$$Z(0) = z_0 \tag{4}$$

where $G(t)$ is a Gaussian white noise characterized by its mean and autocorrelation functions

$$E[G(t)] = 0 \tag{5a}$$

$$E[G(t)G(t+\tau)] = 2\pi S_0 \delta(\tau) \tag{5b}$$

where S_0 is the magnitude of the constant two-sided spectral density function for $G(t)$ and $\delta(\cdot)$ the Dirac delta function. It is well known[22] that $Z(t)$ is a Gaussian random process and that the stationary autocorrelation and spectral density functions of $Z(t)$ are given respectively by

$$R_{zz}(\tau) = \frac{\pi S_0}{\xi} e^{-\xi|\tau|} \tag{6}$$

$$S_{zz}(\omega) = \frac{S_0 \xi}{\xi^2 + \omega^2} \tag{7}$$

The stationary variance of Z is then given by

$$\sigma_z^2 = \pi S_0/\xi \tag{8}$$

Let the random deviate $X(t)$ in Eq. (2) have a marginal distribution function $F_x(x)$. Then, the random process $X(t)$ can be represented by the transformed process[23]

$$X(t) = F_x^{-1}\{\Phi[Z(t)/\sigma_z]\} \tag{9}$$

where $\Phi(\cdot)$ is the standard normal distribution function. Analytical solution does not exist for the distribution function of the random process $A(t)$ when the random deviate $X(t)$ is described as in Eqs. (2) and (9), so numerical solutions are limited to Monte Carlo simulations. In order to study the problem via Markov process theory, Eq. (2) is appended to our crack growth law to obtain the set of first-order stochastic differential equations given by

$$\frac{\mathrm{d}A}{\mathrm{d}t} = H(A)F_x^{-1}\left\{\Phi\left[\frac{Z(t)}{\sigma_z}\right]\right\} \tag{10}$$

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = -\xi Z + G(t) \tag{11}$$

$$A(0) = a_0;\ Z(0) = z_0 \tag{12}$$

where $H(\cdot)$ is a positive definite function of the crack size a. One can show[18] that without resorting to stochastic averaging, Eqs. (10-12) are of the Ito form and that the state vector $[A, Z]'$ is exactly Markovian. Here, the prime denotes transpose of the vector. All of the attendant theory for Markov processes is then available to define the random crack size $A(t)$. Thus, if a solution can be obtained to describe the vectored random process $[A, Z]'$, an appropriate solution can then be obtained for the crack growth problem.

Boundary Value Problem

According to the definition adopted here, the structure will properly perform if the crack size remains less than some critical crack size a_c. Thus, we can define the reliability of the structure as

$$R\ (t/a_0,z_0) = P\{[A(\tau)<a_c,\ 0 \le \tau \le t;\ t/a_0,z_0 \tag{13}$$

The Pontryagin-Vitt equation, a recursive set of boundary value problems for the moments of time to reach a critical crack size a_c, can then be written as[18]

$$-nT^{n-1} = H(a_0)F_x^{-1}\left\{\Phi\left[\frac{z_0}{\sigma_z}\right]\right\}\frac{\partial T^n}{\partial a_0} - \xi z_0 \frac{\partial T^n}{\partial z_0} + \pi S_0 \frac{\partial^2 T^n}{\partial z_0^2} \qquad n = 1,2,\ldots \tag{14}$$

where $T^n = T^n(a_0,z_0)$ is the nth statistical moment of the random time to reach the critical crack size a_c as defined by

$$T^n(a_0,z_0) = -\int_0^\infty t_n \frac{\partial R(t\,|\,a_0,z_0)}{\partial t}\,\mathrm{d}t \tag{15}$$

The initial solution is given by definition as $T^0(a_0,z_0) \equiv 1$ and the boundary conditions are stated on physical grounds as

$$T^n(a_c,z_0) = 0, \qquad \forall z_0 \tag{16a}$$

$$T^n(a_0,z_0) \to 0, \qquad z_0 \to \infty \tag{16b}$$

$$\frac{\partial T^n(a_0,z_0)}{\partial z_0} \to 0, \qquad z_0 \to -\infty \tag{16c}$$

The first boundary condition simply indicates that the moments of time to reach the critical crack size a_c are zero for all z_0 if the initial flaw size is the critical crack size. The second boundary condition is justified by noting that as $z_0 \to \infty$, $\mathrm{d}A/\mathrm{d}t \to \infty$. Thus, the crack will grow very rapidly and, in the limit, the time to reach the critical crack size will go to zero. For the third boundary condition, as $z_0 \to -\infty$, we note that $\mathrm{d}A/\mathrm{d}t \to 0$, and thus the times to reach the critical crack size will become large. The zero slope condition is reached when the time to reach the critical crack size becomes much larger than the correlation time of $Z(t)$ and the effect of the initial condition is minimal. A more thorough discussion of the boundary conditions can be found in Ref. 18. One should also note that, due to the lack of a second derivative in the a_0 direction in Eq. (14), only one boundary condition needs to be specified in the a_0 direction in order that the boundary value problem be well posed.[24] The difficulty encountered in Ref. 11 with respect to prescribing a second boundary condition at $a_0 = 0$ has therefore been avoided. In addition, we have constructed the problem based on a two-dimensional Markov process such that negative crack growth rates are impossible and stochastic averaging is not required. Solution of the above boundary value problem will thus yield the statistical moments of the time to reach a critical crack size conditional on the initial flaw size.

In this problem formulation, the solution is a function of the initial value of the auxiliary random process $Z(t)$. In order to eliminate the dependence on z_0, one can assume a stationary start condition for $Z(t)$ and make use of the theorem of total probability to obtain

$$T(a_0) = \int_{-\infty}^{\infty} T^n(a_0,z_0) f_z(z_0)\, \mathrm{d}z_0 \tag{17}$$

where $f_z(z_0)$ is the stationary probability density function for $Z(t)$. Since $Z(t)$ is Gaussian, the stationary density function is also Gaussian with a zero mean and a variance given by Eq. (8). The integration in Eq. (17) can then be performed numerically.

Numerical Solution

Equation (14) is a two-dimensional convective-transport equation. For such problems, which have significant first partial derivatives relative to their respective second partial derivatives, solution is known to be impossible to obtain analytically and difficult to achieve numerically. The absence of the second partial derivative with respect to a_0 in Eq. (14) causes the equation to be degenerate elliptic-parabolic; as a result, standard Bubnov-Galerkin finite-element methods generally produce unstable solutions. However, a general weighted residual method of the Petrov-Galerkin kind has been developed in Ref. 25 and shown to produce stable and convergent solutions for this class of equations. Before Eq. (14) can be discretized via this finite-element method, however, the infinite boundary conditions given in Eqs. (16b) and (16c) must be addressed. Several methods [26-28] are available to handle infinite boundary conditions; however, the approach employed herein is to place the boundaries at a sufficiently large distance from the domain of interest that the solution therein is not affected. The Petrov-Galerkin finite-element method used in conjunction with this method of handling the infinite boundaries has been shown to be effective for solution of similar problems in Refs. 29–32.

The governing partial differential equation given in Eq. (14) is then cast into the weak form and discretized using unsymmetric biquadratic weighting functions W_i in conjunction with bilinear interpolation functions N_i. Thus, after transforming to isoparametric coordinates (η,λ), the discretized form of Eq. (14) is

$$\boldsymbol{K}\boldsymbol{T}^n = \boldsymbol{B}^{n-1}, \quad n = 1,2,\ldots \tag{18}$$

where

$$k_{ij} = \sum_{N_e} \int_{-1}^{1}\int_{-1}^{1} \left\{ \pi S_0 \frac{\partial W_i}{z_0}\frac{\partial N_j}{z_0} - W_i \left[H(a_0) F_x^{-1} \left\{ \Phi\left(\frac{z_0}{\sigma_z}\right) \right\} \frac{\partial N_j}{\partial a_0} - \xi z_0 \frac{\partial N_j}{\partial z_0} \right] \right\} |J|\, \mathrm{d}\eta \mathrm{d}\lambda \tag{19}$$

$$b_i^{n-1} = \sum_{N_e} \int_{-1}^{1}\int_{-1}^{1} W_i T^{n-1} |J| \mathrm{d}\eta \mathrm{d}\lambda \tag{20}$$

where N_e is the number of elements in the finite-element mesh and $|J|$ the determinant of the Jacobian matrix. This is a system of linear equations that, when solved recursively, yields n statistical moments of the time to reach the critical crack size.

Finally, an important point to note here is that the solution for the moments of the random time to reach a_c is obtained for all values of the initial flaw size a_0. Thus, the use of an initial flaw size distribution can be included in the analysis with little additional effort, since the solution is conditional on a_0. In addition, if crack size is observed upon inspection to be a_i, the updated statistics are simply the conditional statistics evaluated at a_i, $T^n(a_i)$.

Model Validation

Data are essential for validation of any mathematical model. The FCG data reported in Ref. 1 is one of high replication under tightly controlled laboratory conditions and is well suited for validation of probabilistic models such as the one being considered. Replicate tests for 68 identical center cracked specimens were conducted and the a vs n results are depicted in Fig. 1 for values of a = 9–49.8 mm. In order to minimize statistical uncertainty, each of the specimens was taken from a single sheet of 2024-T3 aluminum and each test was performed by the same operator on the same machine. The tests were run at a constant stress range of ΔS = 48.28 MPa and with a stress ratio R = 0.2. This exacting data will be used to validate the stochastic model and to demonstrate its accuracy and versatility.

For the study of FCG under constant-amplitude loading, the crack growth rate is commonly considered a function of the elastic stress intensity range given for a finite-width plate by

$$\Delta K = \Delta S \sqrt{\frac{\pi a}{\cos(\pi a/b)}}, \qquad \frac{a}{b} < 0.7 \tag{21}$$

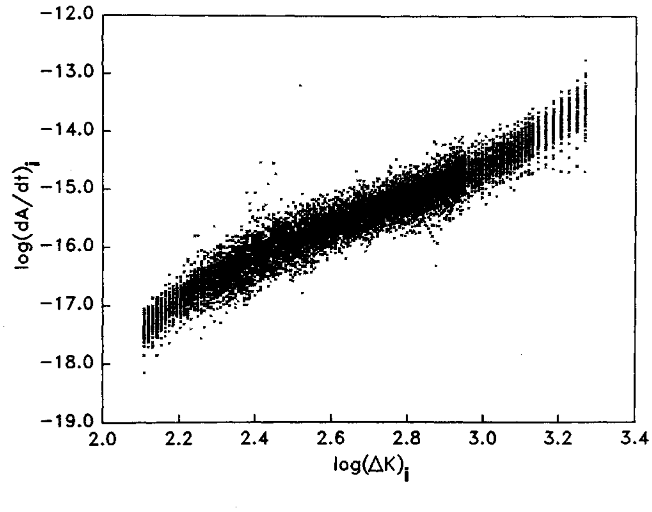

Fig. 2 Log $(\mathrm{d}A/\mathrm{d}t)_i$ vs log $(\Delta K)i$.

where ΔS is the stress range and b the width of the plate. A plot of $\log(\mathrm{d}A/\mathrm{d}t)$ vs $\log(\Delta K)$ for the Virkler data is depicted in Fig. 2, where a linear trend is seen to predominate the behavior.

Paris-Erdogan FCG Law

The commonly used Paris-Erdogan[33] FCG law reflects this linear relationship between $\log(\mathrm{d}a/\mathrm{d}t)$ and $\log(\Delta K)$ and, when randomized, is given by

$$\frac{\mathrm{d}A}{\mathrm{d}t} = C(\Delta K)^m X(t) \tag{22}$$

where C and m are material parameters to be determined below. Several researchers have advocated $X(t)$ being a stationary log normal random process with an exponential autocorrelation function (see Refs. 14, 15, and 21). To examine the Virkler data, we will also use this statistical structure for $X(t)$. While the random process defined by Eqs. (3) and (9) is nonstationary, as discussed above, the dependence of the solution on the initial condition z_0 can be eliminated by using the theorem of total probability and assuming stationary start conditions. Thus, taking $X(t)$ to be a lognormal random process, the transformation in Eq. (3) can be rewritten as

$$X(t) = \exp[Z(t)] \tag{23}$$

The FCG parameters for the Paris-Erdogan law are typically determined by taking the logarithm of both sides of Eq. (22) to obtain

$$\log\left(\frac{\mathrm{d}A}{\mathrm{d}t}\right) = m\log(\Delta K) + \log(C) + Z \tag{24}$$

and using a linear least-square regression analysis to estimate m and $\log(C)$. Here, we see that $Z = \log(X)$ is the random error in the regression analysis. The secant method was employed to determine $\mathrm{d}A/\mathrm{d}t$ for use in the regression analysis and the parameter estimates were found to be

$$C = 8.0958\text{E-}11, \quad m = 2.9123$$

Now, the random residuals Z_i at each data point can be determined via

$$Z_i = \log\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)_i - m\log(\Delta K)_i - \log(C) \tag{25}$$

and are depicted in Fig. 3 where it is readily seen that the residual process is nonstationary. The nonstationarity appears because the Paris-Erdogan model only linearly approximates

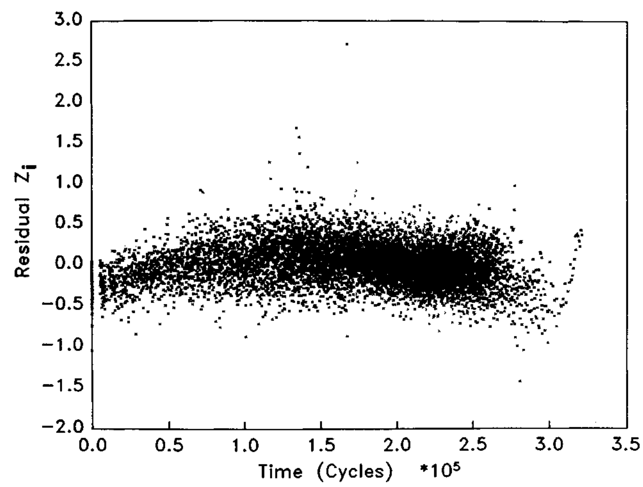

Fig. 3 Residuals for the Paris-Erdogan FCG model.

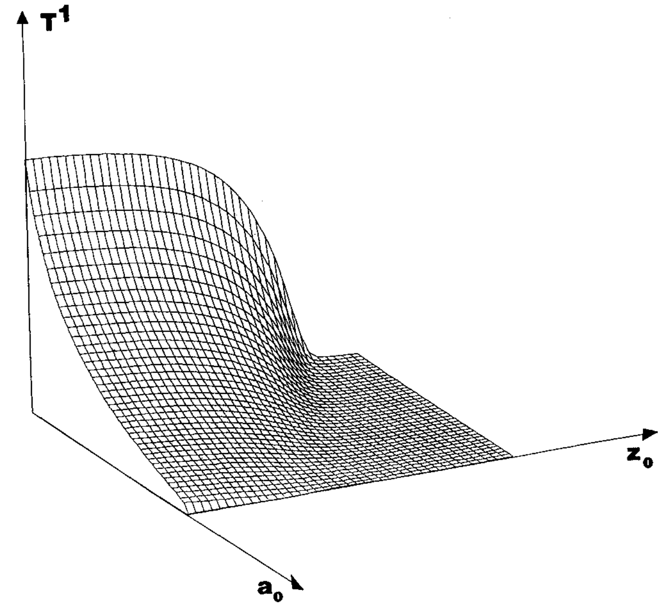

Fig. 4 Finite-element solution for $T^1(a_0,z_0)$, the mean time to reach $a_c = 49.8$ mm as a function initial flaw size and initial value of the auxiliary process $Z(t)$.

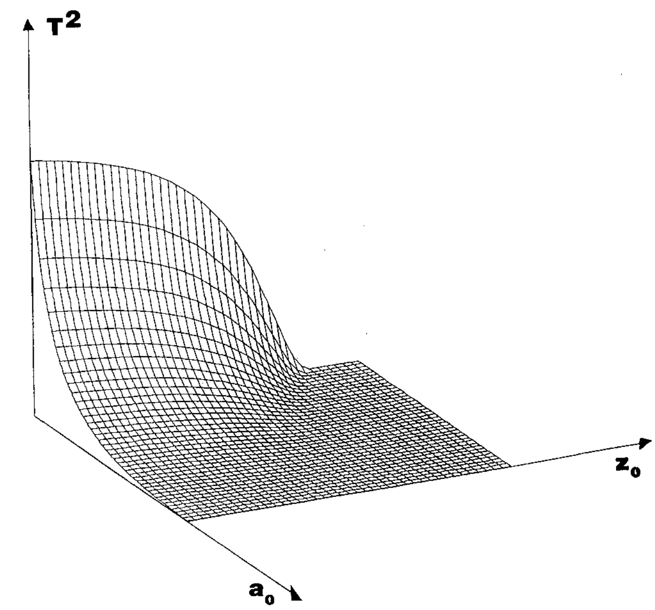

Fig. 5 Finite-element solution for $T^2(a_0,z_0)$, the mean squared time to rach $a_c = 49.8$ mm as a function the initial flaw size and initial value of the auxiliary process $Z(t)$.

the nonlinear behavior of the $\log(\mathrm{d}a/\mathrm{d}t)$ vs $\log(\Delta K)$ data depicted in Fig. 2. Assuming that this nonstationarity can be neglected, the standard deviation of Z can be estimated via

$$\sigma_z^2 = \frac{1}{n-2}\sum_{i=1}^{n} Z_i^2 \tag{26}$$

and is found to be $\sigma_z = 0.23612$. An appropriate value of the correlation parameter is chosen to be $\xi = 10\text{E-}5$. Finally, the deterministic component of the FCG law, $h(a)$ is obtained for the Paris-Erdogan FCG law by substituting ΔK as defined in Eq. (21) into Eq. (22) to find

$$H(a) = C\left[\Delta S\sqrt{\frac{\pi a}{\cos(\pi a/b)}}\right]^m \tag{27}$$

and the finite-element solution for the statistical moments of the time to reach the critical crack size can be computed.

Figures 4 and 5 depict the finite-element solution for the first and second moments of the time to reach a critical crack size of $a_c = 49.8$ mm starting from an initial value of a_0. Here, we see that the surface is smooth and, as we would expect, as the initial flaw size decreases ($a_0 \to 0$), the mean and the mean squared time to reach the critical crack size increase and, as the initial flaw size approaches the critical crack size ($a_0 \to a_c$), the solution decreases to zero. In addition, one can see that the boundary conditions prescribed for z_0 at plus and minus infinity are well represented on the finite domain.

Utilizing Eq. (17), the dependence of the solution depicted in Fig. 4 on z_0 can be eliminated and the sought statistics are obtained. Figure 6 depicts the finite-element solution for the mean time to reach several critical crack sizes conditional on $a_0 = 9$ mm for the Paris-Erdogan model. The Virkler data are superimposed for comparison. As can be seen, the results are biased toward faster growing fatigue cracks. This phenomena has been recognized by other researchers in deterministic crack growth analyses and is generally attributed to either the numerical differentiation or the nonlinear nature of the data. However, a finite-integral method will be introduced and modified to alleviate this problem and produce an unbiased estimate of the parameters required for the stochastic FCG analysis.

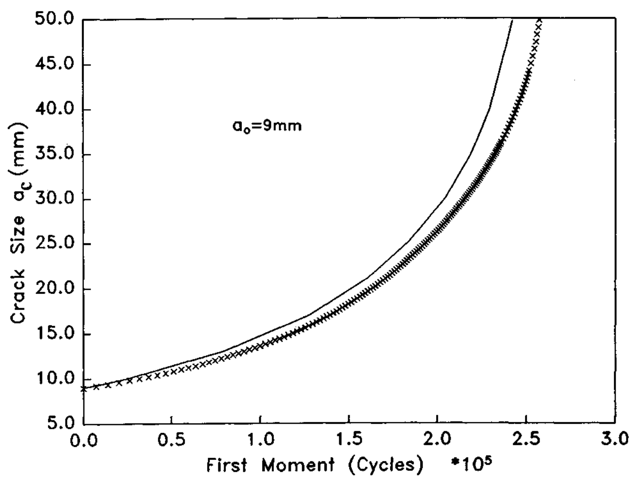

Fig. 6 Comparison between the predicted mean time to reach a_c and the experimental dta for $a_0 = 9$ mm (Paris-Erdogan FCG law using linear regression).

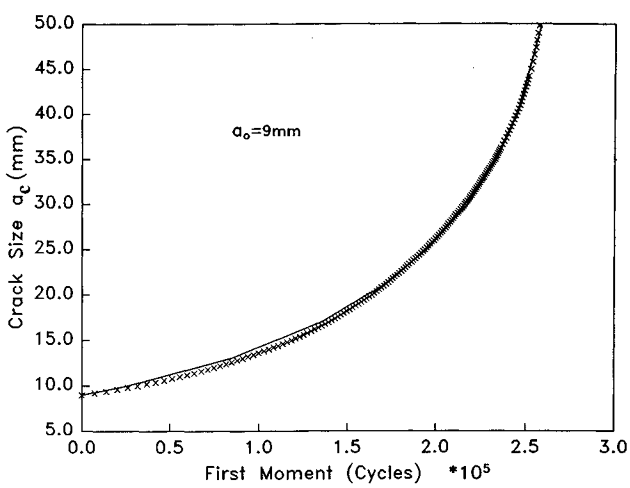

Fig. 7 Comparison between the predicted mean time to reach a_c and the experimental data for $a_0 = 9$ mm (Paris-Erdogan FCG law using the MFI method).

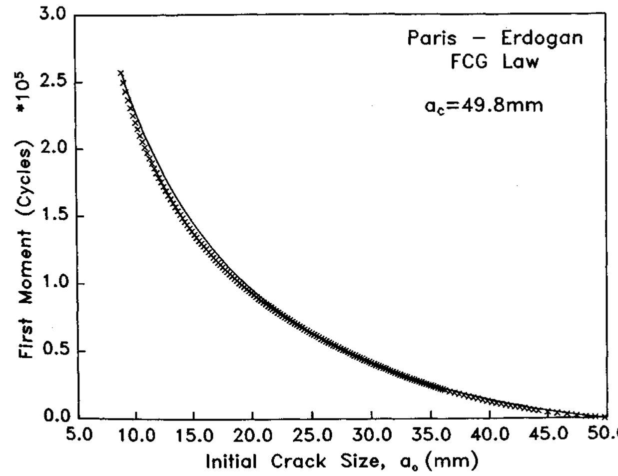

Fig. 8 Comparison between the predicted mean time to reach $a_c = 49.8$ mm and the experimental data (Paris-Erdogan FCG law using the MFI method).

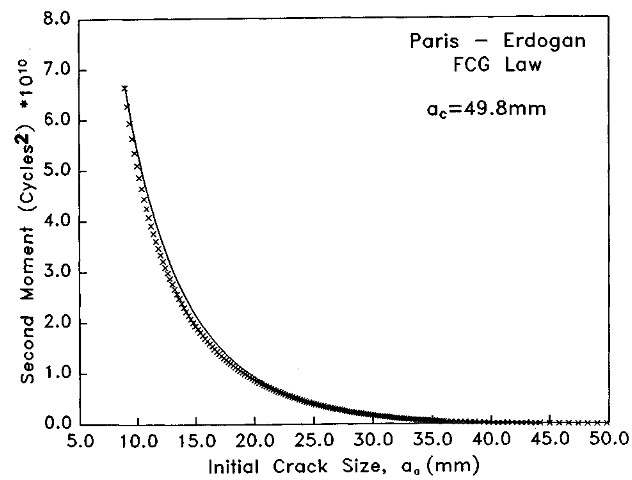

Fig. 9 Comparison between the predicted mean squared time to reach $a_c = 49.8$ mm and the experimental data (Paris-Erdogan FCG law using the MFI method).

Modified Finite-Integral Method

Ostergaard and Hillberry[34] have suggested a finite-integral technique for producing unbiased estimates of the parameters in deterministic FCG laws. This technique is modified herein for parameter estimation in stochastic FCG analysis. Here, the parameters of the stochastic FCG law are separated, integrated, and averaged over the ensemble to obtain

$$\int_{a_0}^{a_i} \frac{d\varphi}{H(\varphi)} = E\left[\int_0^{T_{r_i}} X(t)\ dt\right] \approx \mu_x E[T_{r_i}] \tag{28}$$

where a_i is specified and μ_x is the stationary mean of $X(t)$. Equation (28) is numerically integrated through finite crack length levels. For the Virkler data, we integrated at 51 constant Δa increments of 0.8 mm and defined an error term as

$$e = \sum_{i=1}^{51} [R(T_{a_i}) - E(T_{r_i})]^2 \tag{29}$$

where T_i is the cumulative cycle count from the Virkler data and T_{r_i} the cumulative cycle count from the identical integration. The optimum FCG law parameters are those that produce the minimum error as defined in Eq. (29) and can be obtained using the IMSL routine ZXSSQ.[35]

The normalized error at a given point on the integrated a vs t curves resulting from use of these parameters can then be defined as

$$\epsilon_i = \sqrt{e}/51 \tag{30}$$

Thus, the magnitude of ϵ_i can be used as an indicator of the quality of a model in representing the FCG phenomena, with the best models producing the smallest ϵ_i. The optimum FCG parameters for the Paris-Erdogan law are then determined to be

$$C = 8.3110\text{E-}11, \qquad m = 2.8749$$

and the normalized error is $\epsilon_i = 439$.

Differentiation of the FCG data is not required in the MFI method to obtain the material constants; however, it is required to obtain the random residuals Z_i defined in Eq. (25). The standard deviation of Z is thus found to be $\sigma_z = 0.24632$ and the correlation parameter is again chosen to be $\xi = 10\text{E-}5$.

Figure 7 depicts the finite-element solution for the mean time to reach several critical crack sizes conditional on $a_0 = 9$ mm for the Paris-Erdogan model. The Virkler data are superimposed for comparison and, as can be seen, the MFI method produces unbiased estimates of the parameters. The slight deviations from the data found at lower values of a_c can be eliminated through using a better FCG law as demonstrated below. Several finite-element runs were made to obtain Fig. 7; however, each finite-element solution is obtained for all values of the initial flaw size a_0. Thus, Figs. 8–10 were obtained from one run of the finite-element program. In these figures, the mean, mean square, and standard deviation, respectively, of the time to reach a critical crack size of 49.8 mm are depicted for the Paris-Erdogan model as a function of a_0. The Virkler data are again superimposed for comparison. Good correlation with experimental data is found, with the finite-element solution being slightly conservative for smaller initial flaw sizes. This is due to Paris-Erdogan FCG law being linear and can be alleviated by employing a more appropriate FCG law.

Cubic Polynomial FCG Law

Although the linear Paris-Erdogan relationship does at first appear to be a reasonable fit of the log($\mathrm{d}a/\mathrm{d}t$) vs log(ΔK) data, nonlinear behavior is noted for extreme values of log(ΔK) in Fig. 2. Thus, FCG relationships that model this nonlinear behavior have been proposed. A recent paper by Ortiz and Kung[36] that defines a cubic polynomial in terms of the log(ΔK) is suggested for the FCG law and reads

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \exp[\beta_0 + \beta_1 (\log\Delta K) + \beta_2(\log\Delta K)^2 + \beta_3(\log\Delta K)^3]\, X(t) \tag{31}$$

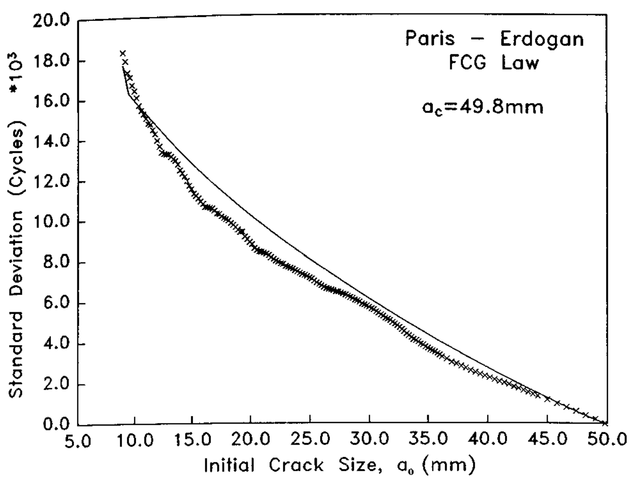

Fig. 10 Comparison between the predicted standard deviation of the time to reach $a_c = 49.8$ mm and the experimental data (Paris-Erdogan FCG law using the MFI method).

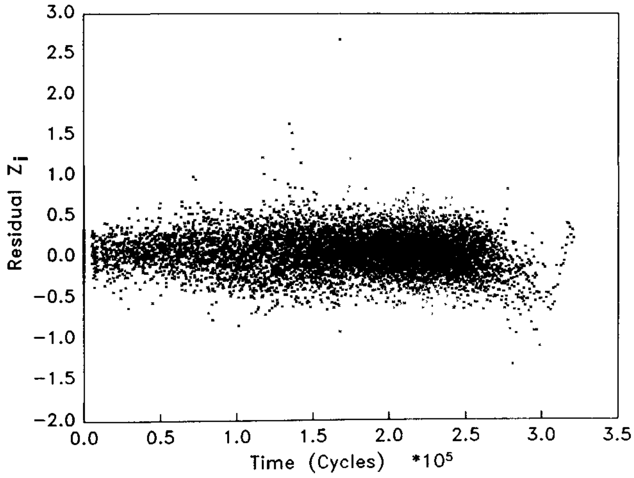

Fig. 11 Residuals for the cubic polynomial FCG model.

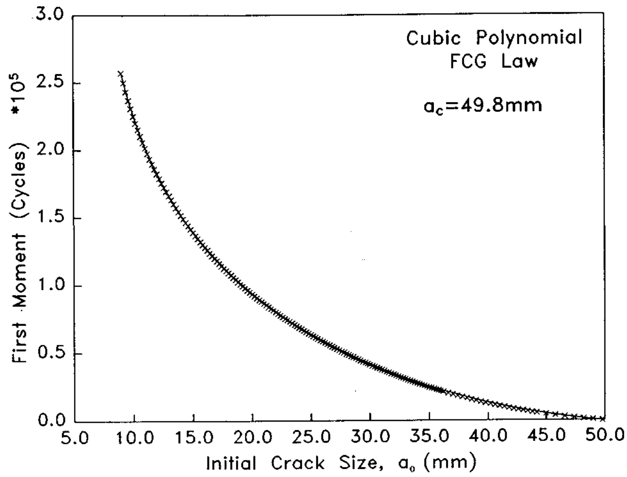

Fig. 12 Comparison between the predicted mean time to reach $a_c = 49.8$ mm and the experimental data (cubic polynomial FCG law using the MFI method).

The parameters determined using the MFI approach are

$$\beta_0 = -74.086, \quad \beta_1 = 60.276, \quad \beta_2 = 21.390, \quad \text{and} \quad \beta_3 = 2.6348$$

and the normalized error has now been reduced by an order of magnitude to obtain $\epsilon_i = 30$. The random residuals Z_i for this model are depicted in Fig. 11. The standard deviation of Z is found to be $\sigma_z = 0.22706$ and the correlation parameter is chosen to be $\xi = 9\text{E-}5$. We see in Fig. 11 that by employing this nonlinear model, the systematic error found in the Paris-Erdogan model has been removed.

The finite-element solutions employing the cubic polynomial FCG model are very similar to those depicted in Figs. 4 and 5 and will be omitted for brevity. Figures 12–14 present results for the cubic polynomial model and parallel Figs. 8–10 for the Paris-Erdogan FCG law. Here we see that the finite-element solution for the mean and mean squared time to reach a_c is indiscernible from the experimental data, even for small values of a_0, and that the solution for the standard deviation has improved slightly over the previous result.

The probability of reaching a critical crack size can also be obtained by fitting statistical moments obtained by the finite-element method to a probability distribution function. Kozin and Bogdanoff[10] have indicated that an appropriate distribution is a type I asymptotic largest value probability distribution given by

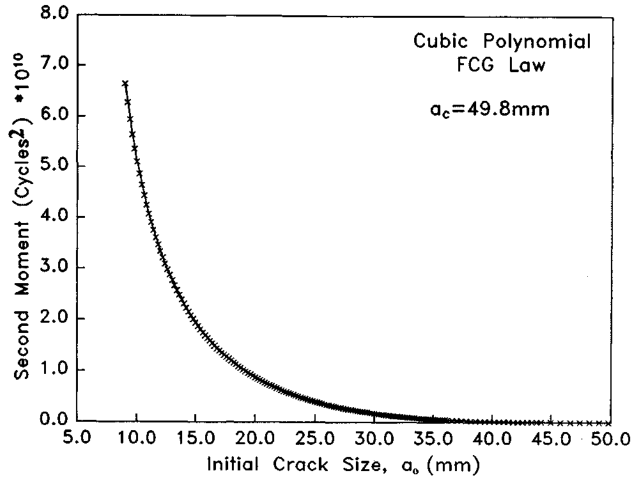

Fig. 13 Comparison between the predicted mean squared time to reach $a_c = 49.8$ mm and the experimental data (cubic polynomial FCG law using the MFI method).

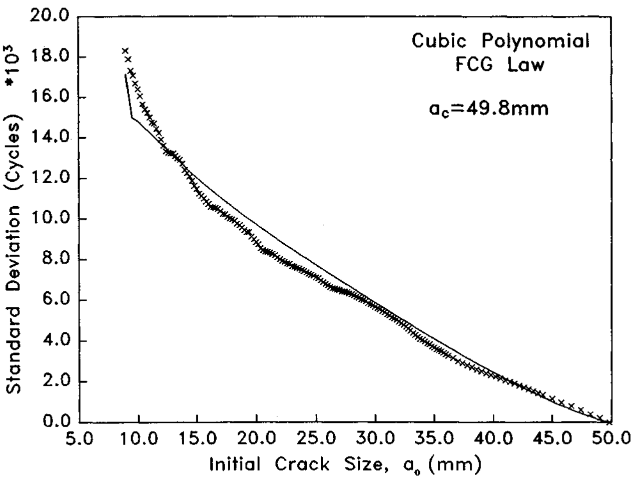

Fig. 14 Comparison between the predicted standard deviation of the time to reach $a_c = 49.8$ mm and the experimental data (cubic polynomial FCG law using the MFI method).

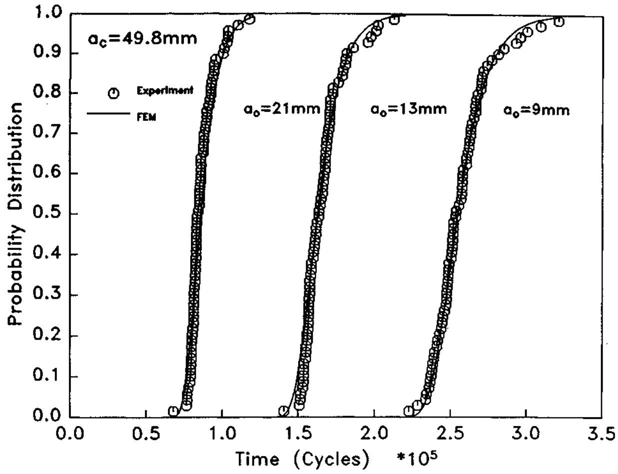

Fig. 15 Comparison between type I asymptotic largest approximation for the distribution of random time to reach $a_c = 49.8$ mm and experimental data ($a_0 = 9$, 13, and 21 mm).

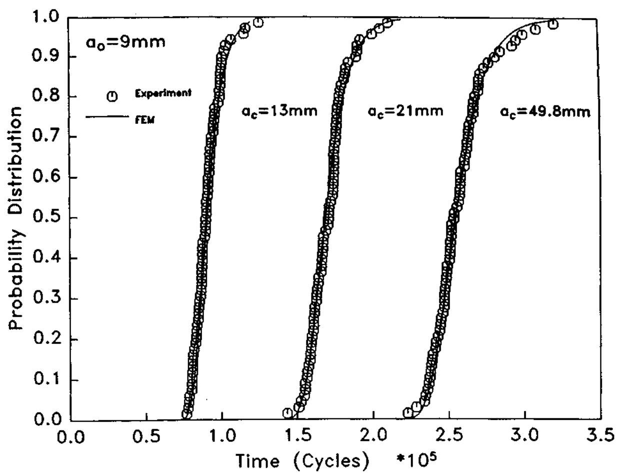

Fig. 16 Comparison between type I asymptotic largest approximation for the distribution of random time to reach $a_c = 13, 21, 49.8$ mm, respectively, and experimental data ($a_0 = 9$ mm).

$$F_T(t) = \exp\{\exp[-\alpha(t-u)]\} \tag{32}$$

The parameters of the distribution are defined in terms of the statistical moments of the time to reach the critical crack size by

$$\alpha = \pi/\sqrt{6\{T^2(a_0) - [T^1(a_0)]^2\}} \tag{33}$$

$$u = T^1 - 0.5772/\alpha \tag{34}$$

Figures 15 and 16 give a comparison between the type I largest value distribution fitted to the finite-element results and the experimental data. Figure 15 presents the results for the distribution of random times to reach the critical crack size for $a_0 = 9$, 13, and 21 mm. In contrast, Fig. 16 depicts the solution for a fixed initial flaw size of $a_0 = 9$ mm with $a_c = 13$, 21, and 49.8 mm. Again, excellent agreement is found for all cases, especially in the tails of the distribution.

Conclusions

A stochastic model for fatigue crack propagation has been investigated in which a two-dimensional state vector has been employed to introduce the experimentally observed history dependence of fatigue crack growth. Comparison between the proposed model and an extensive set of experimental data has been made employing both the Paris-Erdogan and the cubic polynomial fatigue crack growth laws. A modified finite-integral (MFI) approach has been employed to estimate several of the parameters in the fatigue crack growth laws. A well-posed boundary value problem was detailed and a robust Petrov-Galerkin finite-element method was used for solution.

The finite-element solution for the first and second statistical moments of random time to reach a critical crack size conditional on the initial condition a_0 was obtained for both the Paris-Erdogan and the cubic polynomial FCG laws. Linear regression was shown to be unsuitable for estimating model parameters of the Paris-Erdogan model for use in the stochastic FCG analysis, so a MFI method was introduced that resulted in unbiased estimates of the parameters. The finite-element solutions were in good agreement with the experimental data obtained by Virkler et al.,[1] with the first and second moments being indiscernible from the data when the cubic polynomial FCG law was employed.

The probability of reaching a critical crack size was obtained using a type I largest value probability distribution and the finite-element solution. Excellent agreement was found between the numerical solution and the experimental distribution of times to reach the critical crack. The capability of this stochastic approach, over previous ones, is especially evident in the tails of these distributions, which closely mirror the data over the entire range of crack sizes.

Finally, one should note that the solution is obtained for all physically realizable values of initial flaw size and, thus, only one finite-element solution was required to obtain the mean time to reach a_c as a function of a_0. Thus, the effect of initial fatigue quality is readily assessed through the assumption of an appropriate initial flaw size distribution and application of the theorem of total probability. In addition, in the solution for the higher moments, the stiffness matrix does not need to be refactored. The successive statistical moments require only calculation of the new right-hand side from the previous solution and subsequent forward and backward substitution. This is, for two or more moments, an extremely cost-effective calculation when one considers the amount of information to be gained in a single analysis.

Acknowledgment

The authors gratefully acknowledge Prof. B. M. Hillberry for providing the Virkler data used in this study.

References

[1]Virkler, D. A., Hillberry, B. M., and Goel, P. K., "The Statistical Nature of Fatigue Crack Propagation," *Journal of Engineering Materials and Technology*, Vol. 101, 1979, pp. 148–152.

[2]Manning, S. D. et al., "Durability Methods Development, Vol. I, Phase 1 Summary," Air Force Flight Dynamics Laboratory, AFFDL-TR-79-3118, Sept. 1979.

[3]MIL-A-87221, "General Specification for Aircraft Structures," Feb. 1985, pp. 377–386.

[4]Manning, S. D. and Yang, J. N., "Advanced Durability Analysis, Volume I – Analytical Methods," Air Force Wright Aeronautical Laboratories, AFWAL-TR-86-3017, July 1987.

[5]Manning, S. D. and Yang, J. N., "USAF Durability Design Handbook: Guidelines for the Analysis and Design of Durable Aircraft Structures," Air Force Wright Aeronautical Laboratories, AFWAL-TR-83-3027, Jan. 1984.

[6]Miller, M. S. and Gallagher, J. P., "An Analysis of Several Fatigue Crack Growth Rate (FCGR) Descriptions," *Fatigue Crack Growth Measurement and Data Analysis*, STP 738, American Society for Testing and Materials, Philadelphia, 1981, pp. 205–251.

[7]Bogdanoff, J. L. and Kozin, F., *Probabilistic Models of Cumulative Damage*, Wiley, New York, 1985.

[8]Ditlevsen, O. and Olesen, R., "Random Fatigue Crack Growth—A First Passage Problem," *Engineering Fracture Mechanics*, Vol. 23, 1986, pp. 476–477.

[9]Ditlevsen, O. and Olesen, R., "Statistical Analysis of the Virkler Data on Fatigue Crack Growth," *Engineering Fracture Mechanics,* Vol. 25, 1986, pp. 177–195.

[10]Kozin, F. and Bogdanoff, J. L., "Probabilistic Models of Fatigue Crack Growth: Results and Speculations," *Journal of Nuclear Engineering and Design*, to be published.

[11]Lin, Y. K. and Yang, J. N., "On Statistical Moments of Fatigue Crack Propagation," *Journal of Engineering Fracture Mechanics*, Vol. 18, No. 2, 1983, pp. 243–256.

[12]Lin, Y. K. and Yang, J. N., "A Stochastic Theory of Fatigue Crack Propagation," *AIAA Journal*, Vol. 1, Jan. 1985, pp. 117–124.

[13]Lin, Y. K., Wu, W. F., and Yang, J. N., "Stochastic Modeling of Fatigue Crack Propagation," *Proceedings of the IUTAM Symposium on Probabilistic Methods in Mechanics of Solids and Structures*, June 1984, pp. 103–110.

[14]Ortiz, K. and Kiremidjian, A. S., "Time Series Analysis of Fatigue Crack Growth Data," *Engineering Fracture Mechanics*, Vol. 24, 1986, pp. 657–676.

[15]Ortiz, K. and Kiremidjian, A. S., "Stochastic Modeling of Fatigue Crack Growth," *Engineering Fracture Mechanics*, to be published.

[16]Palmberg, B., Blom, A. F., and Eggwertz, S., "Probabilistic Damage Tolerance Analysis of Aircraft Structures," *Aeronautical Research Institute of Sweden*, Tech. Rept. FFA TN 1985-68, 1985; also *Probabilistic Fracture Mechanics and Reliability*, edited by J. Provan, Martinus Nijhoff, the Netherlands, 1987.

[17]Sobcyzk, K., "Stochastic Modeling of Fatigue Crack Growth," *Proceedings of IUTAM Symposium on Probabilistic Methods in Mechanics of Solids and Structures*, June 1984, pp. 111–119.

[18]Spencer, B. F., Jr. and Tang, J., "A Markov Process Model for Fatigue Crack Growth," *Journal of Engineering Mechanics*, to be published.

[19]Yang, J. N., Salivar, G. C., and Annis, C. G., "Statistical Modeling of Fatigue-Crack Growth in a Nickel-Based Superalloy," *Journal of Engineering Fracture Mechanics*, Vol. 18, No. 2, 1983, pp. 257–279.

[20]Yang, J. N. and Donath, R. C., "Statistical Crack Propagation in Fastener Holes under Spectrum Loading," *Journal of Aircraft*, Vol. 20, 1983, pp. 1028–1032.

[21]Yang, J. N., Hsi, W. H., and Manning, S. D., "Stochastic Crack Propagation with Application to Durability and Damage Tolerance Analyses," Air Force Wright Aeronautical Laboratories, AFWAL-TR-85-3062, 1985; also, *Probabilistic Fracture Mechanics and Reliability*, edited by J. Provan, Martinus Nijhoff, the Netherlands, 1987.

[22]Lin, Y. K., *Probabilistic Theory of Structural Dynamics*, McGraw-Hill, New York, reprinted by R. E. Krieger, New York.

[23]Grigoriu, M., "Crossings of Non-Gaussian Translation Processes," *Journal of Engineering Mechanics*, Vol. 110, No. 4, 1984, pp. 610–620.

[24]Fichera, G., "On a Unified Theory of Boundary Value Problems for Elliptic-Parabolic Equations of Second Order," *Boundary Problems in Differential Equations*, edited by R. E. Langer, University of Wisconsin, Madison, WI, 1960, pp. 131–143.

[25]Heinrich, J. C., Huyakorn, P. S., Zienkiewicz, O. C., and Mitchell, A. R., "An 'Upwind' Finite Element Scheme for Two Dimensional Convective Transport Equation," *International Journal for Numerical Methods in Engineering*, Vol. 11, 1977, pp. 131–143.

[26]Bettess, P. and Zienkiewicz, O. C., "Diffraction and Refraction of Surface Waves Using Finite and Infinite Elements," *International Journal of Numerical Methods in Engineering*, Vol. 11, 1977, pp. 1271–1290.

[27]Zienkiewicz, O. C., Kelly, D. W., and Bettess, P., "The Coupling of the Finite Element Method and Boundary Solution Procedures," *International Journal of Numerical Methods in Engneering*, Vol. 11, 1977, pp. 355–375.

[28]Zienkiewicz, O. C. and Chung, Y. K., "Buttress Dams on Complex Rock Foundations," *Water Power*, Vol. 15, 1984, p. 193.

[29]Bergman, L. A. and Heinrich, J. C., "Solution of the Pontriagin-Vitt Equation for the Moments of Time to First Passage of the Randomly Accelerated Particle by the Finite Element Method," *International Journal for Numerical Methods in Engineering*, Vol. 14, 1980, pp. 1408–1412.

[30]Bergmann, L. A. and Heinrich, J. C., "On Moments of Time to First Passage of the Linear Oscillator," *Earthquake Engineering and Structural Dynamics*, Vol. 9, 1981, pp. 197–204.

[31]Spencer, B. F., Jr. and Bergmann, L. A., "On the Reliability of a Simple Hysteretic System," *Journal of Engineering Mechanics*, Vol. 111, No. 12, 1985, pp. 1502–1514.

[32]Spencer, B. F., Jr., "On the Reliability of Nonlinear Hysteretic Structures Subjected to Broadband Random Excitation," *Lecture Notes in Engineering*, Vol. 21, Springer-Verlag, New York, 1986.

[33]Paris, P. C. and Erdogan, F., "A Critical Analysis of Crack Propagation Laws," *Transactions of ASME, Journal of Basic Engineering*, Vol. 85, 1963, pp. 528–534.

[34]Ostergaard, D. F. and Hillberry, B. M., "Characterization of the Variability in Fatigue Crack Propagation Data," *Probabilistic Methods for Design and Maintenance of Structures*, STP 798, American Society for Testing and Materials, Philadelphia, PA, 1983.

[35]"IMSL, The International Math and Statistics Subroutine Library," IMSL Inc., Houston, TX, 1984.

[36]Ortiz K. and Kung, C. J., "Modeling Errors in Fatigue Crack Growth Laws," *Materials and Member Behavior*, 1987.